GREEN FUNCTIONS FOR TRANSPORT PROBLEMS IN TIME-VARYING MEDIA

V. S. Novikov

UDC 519.25:536.2.001:538.56

The Green functions are formulated for boundary-value problems associated with the phenomenological parabolic transport equation with time-dependent coefficients.

We have previously [1] formulated the Green function for the mixed Dirichlet–Cauchy transport boundary-value problem described by a certain one-dimensional unsteady-state parabolic equation with coefficients in the form of arbitrary time functions. The formulation was based on the simultaneous application of Chandrasekhar's method [2] to find the fundamental solution of the given equation, along with the method of superposition of sources and sinks [3, 4].

Below, we formulate the Green functions of one-dimensional transport problems for an equation representing a generalization of the equation in [1] and having the form

$$\frac{\partial T}{\partial t} = \varphi^2(t)\frac{\partial^2 T}{\partial x^2} + 2f(t)\frac{\partial T}{\partial x} + F(t)\,T. \tag{1}$$

We formulate the fundamental solution of Eq. (1), which has a single source at $x = x_0$ for $t = t_0$. Following Chandrasekhar [2], we seek this solution in the form

$$\Gamma(x,\ t;\ x_0,\ t_0) = \frac{1}{2\pi\nu^{-\frac{1}{2}}}\exp\left[-\frac{a_1(x-x_0)^2 + 2h_1(x-x_0)}{2\nu}\right], \tag{2}$$

where the functions $\nu = \nu(t)$, $a_1 = a_1(t)$, $h_1 = h_1(t)$ are to be determined. We denote $a_1/\nu = a$, $h_1/\nu = h$ and substitute expression (2) into (1). Equating the coefficients of like powers of $x - x_0$ in the resulting expression, we obtain the system of equations

$$-\frac{1}{2}\frac{da}{dt} = \varphi^2 a^2,\quad -\frac{1}{2}\frac{dh}{dt} = \varphi^2 ah - af,\quad -\frac{1}{2}\frac{1}{\nu}\frac{d\nu}{dt} = -2fh - a\varphi^2 + \varphi^2 h^2 + F. \tag{3}$$

It follows from (3) that

$$a(t) = \frac{1}{2\int\varphi^2 dt + C},\quad h(t) = \frac{2\int f dt}{2\int\varphi^2 dt + C} + \frac{B}{A(2\int\varphi^2 dt + C)}, \tag{4}$$

$$\nu(t) = D\exp\left[2\int(2fh + a\varphi^2 - \varphi^2 h^2 - F)\,dt\right]. \tag{5}$$

We substitute these functions into (2) and determine the constants of integration A, B, C, and D from the condition that for $t \to t_0$ and $x \to x_0$ expression (2) has the properties of a delta function. Inasmuch as the function Γ must go over to a delta function and tend to infinity as $t \to t_0$ and $x \to x_0$, we infer that $C = 0$, $B/A = 0$, and $D = 1/2\pi$ if the medium is assumed to move initially with a velocity $2f(t)$ at the time $t = t_0$. In this case the integrals are evaluated between the limits t_0 and t, so that

$$\Gamma(x,\ t;\ x_0,\ t_0) = \frac{\Phi(t,\ t_0)}{2\pi\left[2D\Psi(t,\ t_0)\right]^{\frac{1}{2}}}\exp\left[-\frac{(x - x_0 + \omega(t,\ t_0))^2}{4\Psi(t,\ t_0)}\right],$$

$$\Phi(t,\ t_0) = \exp\left[\int_{t_0}^{t}\varphi^2(t)\left(\frac{\int f(t)\,dt}{\int\varphi^2(t)\,dt}\right)^2 dt - 2\int_{t_0}^{t}\frac{f(t)\left(\int f(t)\,dt\right)}{\int\varphi^2(t)\,dt}\,dt + \right. \tag{6}$$

Institute of Technical Thermophysics, Academy of Sciences of the Ukrainian SSR, Kiev.
Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 42, No. 6, pp. 1020-1023, June, 1982. Original article submitted December 31, 1980.

0022-0841/82/4206-0699$07.50

$$+\frac{\omega^2(t,\ t_0)}{4\Psi(t,\ t_0)}+\int_{t_0}^{t} F(t)\,dt\Bigg],\quad \omega(t,\ t_0)=2\int_{t_0}^{t} f(t)\,dt,\quad \Psi(t,\ t_0)=\int_{t_0}^{t}\varphi^2(t)\,dt.$$

If we set f(t) = 0, F(t) = 0, $\varphi^2(t)$ = const, expression (6) goes over to the fundamental solution of the heat-conduction equation.

To find the Green functions for specific boundary-value problems, we use the method of superposition of sources and sinks [3, 4], the basic idea of which is to arrange δ-function heat (mass) sources and sinks of unit strength outside the limits of the investigated one-dimensional region in accordance with the type of boundary-value problem in such a way as to ensure fulfillment of the given boundary conditions. The essential details of the method are described in [3, 4], so that in the ensuing discussion we omit the intermediate calculations and arguments and give only the final results. In the interval (0, R) the Green functions for the Dirichlet, Neumann, and all mixed (with inclusion of the Cauchy) boundary-value problems for Eq. (1) have the form

$$G(x,\ t;\ x_0,\ t_0)=\frac{\Phi(t,\ t_0)}{2[\pi\Psi(t,\ t_0)]^{\frac{1}{2}}}\sum_{n=-\infty}^{\infty}\Bigg\{(-1)^k\exp\left[-\frac{(x-x_0+\omega(t,\ t_0)-2nR)^2}{4\Psi(t,\ t_0)}\right]+$$
$$+(-1)^l\exp\left[-\frac{(x+x_0-\omega(t,\ t_0)-2nR)^2}{4\Psi(t,\ t_0)}\right]\Bigg\}+$$
$$+\frac{\Phi(t,\ t_0)}{2[\pi\Psi(t,\ t_0)]^{\frac{1}{2}}}\sum_{|n|=r}^{\infty}\sum_{r=0}^{\infty}(2\alpha)^{r+1}\Bigg\{(-1)^{k+r+1}(|n|-r)\int_0^\infty\exp(-\alpha\eta_1)\ldots$$
$$\ldots\int_0^\infty\exp(-\alpha\eta_r)\int_0^\infty\exp\left[-\alpha\eta_{r+1}-\frac{(x-x_0+\omega(t,\ t_0)-2nR+\sum_{i=1}^{r+1}\eta_i)^2}{4\Psi(t,\ t_0)}\right]d\eta_1\ldots d\eta_{r+1}+$$
$$+(-1)^{l+r+1}(|n+p|-r)\int_0^\infty\exp(-\alpha\eta_1)\ldots\int_0^\infty\exp(-\alpha\eta_r)\times$$
$$\times\int_0^\infty\exp\left[-\alpha\eta_{r+1}-\frac{(x+x_0-\omega(t,\ t_0)-2nR+\sum_{i=1}^{r+1}\eta_i)^2}{4\Psi(t,\ t_0)}\right]d\eta_1\ldots d\eta_{r+1}\Bigg\}\qquad(7)$$

Here α ≡ α(t) is an arbitrary time function; k = 0, l = 1, α = 0 for the Dirichlet problem ($G|_{x=0}$ = 0); k = 0, l = 0, α = 0 for the Neumann problem $\left(\left.\frac{\partial G}{\partial x}\right|_{x=0,R}=0\right)$; $k=n,\ l=n+1,\ \alpha=0$ for the mixed Dirichlet–Neumann problem $\left(G|_{x=0}=\left.\frac{\partial G}{\partial x}\right|_{x=R}=0\right)$; $k=n,\ l=n,\ \alpha=0$ for the mixed Neumann–Dirichlet problem $\left(\left.\frac{\partial G}{\partial x}\right|_{x=0}=G|_{x=R}=0\right)$; $k=n-1,\ l=n+1,\ p=0$ for the mixed Dirichlet–Cauchy problem $\left(G|_{x=0}=\left(\frac{\partial G}{\partial x}+\alpha G\right)_{x=R}=0\right)$; $k=n,\ l=n,\ p=-1$ for the mixed Cauchy–Dirichlet problem $\left(\left(\frac{\partial G}{\partial x}+\alpha G\right)_{x=0}=G|_{x=R}=0\right)$; $k=0,\ l=0,\ p=0$ for the mixed Neumann–Cauchy problem $\left(\left.\frac{\partial G}{\partial x}\right|_{x=0}=\left(\frac{\partial G}{\partial x}+\alpha G\right)_{x=R}=0\right)$; $k=0,\ l=0,\ p=-1$ for the mixed Cauchy–Neumann problem $\left(\left(\frac{\partial G}{\partial x}+\alpha G\right)_{x=0}=\left.\frac{\partial G}{\partial x}\right|_{x=R}=0\right)$.

For small values of α(t) expression (7) is rewritten in the form

$$G(x,\ t;\ x_0,\ t_0)=\frac{\Phi(t,\ t_0)}{2[\pi\Psi(t,\ t_0)]^{\frac{1}{2}}}\sum_{n=-\infty}^{\infty}\Bigg\{(-1)^k\exp\left[-\frac{(x-x_0+\omega(t,\ t_0)-2nR)^2}{4\Psi(t,\ t_0)}\right]+$$

$$+(-1)^{l}\exp\left[-\frac{(x+x_0-\omega(t,\ t_0)-2nR)^2}{4\Psi(t,\ t_0)}\right]+ \tag{8}$$

$$+(-1)^{k+1}2\alpha|n|\int_0^\infty \exp\left[-\alpha\eta-\frac{(x-x_0+\omega(t,\ t_0)-2nR+\eta)^2}{4\Psi(t,\ t_0)}\right]d\eta+$$

$$+(-1)^{l+1}2\alpha|n+p|\int_0^\infty \exp\left[-\alpha\eta-\frac{(x+x_0-\omega(t,\ t_0)-2nR+\eta)^2}{4\Psi(t,\ t_0)}\right]d\eta\Bigg\}.$$

If we confine the discussion to terms containing $\alpha_j(t)$ in the first power, the Green function for the Cauchy problem $\left(\left(\frac{\partial G}{\partial x}+\alpha_1 G\right)_{x=0}=\left(\frac{\partial G}{\partial x}+\alpha_2 G\right)_{x=R}=0\right)$ has the form

$$G(x,\ t;\ x_0,\ t_0)=\frac{\Phi(t,\ t_0)}{2[\pi\Psi(t,\ t_0)]^{\frac{1}{2}}}\sum_{n=-\infty}^{\infty}\left\{\exp\left[-\frac{(x-x_0+\omega(t,\ t_0)-2nR)^2}{4\Psi(t,\ t_0)}\right]+\right.$$

$$\left.+\exp\left[-\frac{(x+x_0-\omega(t,\ t_0)-2nR)^2}{4\Psi(t,\ t_0)}\right]\right\}-\frac{\Phi(t,\ t_0)}{2[\pi\Psi(t,\ t_0)]^{\frac{1}{2}}}\times$$

$$\times\sum_{n=-\infty}^{\infty}|n|\sum_{j=1}^{\infty}\alpha_j\left\{\int_0^\infty \exp\left[-\alpha_j\eta-\frac{(x-x_0+\omega(t,\ t_0)-2nR+\eta)^2}{4\Psi(t,\ t_0)}\right]d\eta+\right. \tag{9}$$

$$\left.+\int_0^\infty \exp\left[-\alpha_j\eta-\frac{(x+x_0-\omega(t,\ t_0)-2nR+\eta)^2}{4\Psi(t,\ t_0)}\right]d\eta\right\}+$$

$$+\sum_{n=0}^{\infty}2\left\{\alpha_1\int_0^\infty \exp\left[-\alpha_1\eta-\frac{(x-x_0+\omega(t,\ t_0)-2nR+\eta)^2}{4\Psi(t,\ t_0)}\right]d\eta+\right.$$

$$\left.+\alpha_2\int_0^\infty \exp\left[-\alpha_2\eta-\frac{(x+x_0-\omega(t,\ t_0)-2nR+\eta)^2}{4\Psi(t,\ t_0)}\right]d\eta\right\}.$$

The terms containing $\alpha_j(t)$ in higher than the first power are extremely cumbersome.

LITERATURE CITED

1. V. S. Novikov, "Approximate analysis of transport processes in clouds," Izv. Akad. Nauk SSSR, Fiz. Atm. Okeana, 14, No. 12, 123-124 (1978).
2. S. Chandrasekhar, Stochastic Problems in Physics and Astronomy [Russian translation], IL, Moscow (1947).
3. G. N. Polozhii, Equations of Mathematical Physics [in Russian], Vysshaya Shkola, Moscow (1964).
4. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, 2nd ed., Clarendon, Oxford (1959).